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On local comparison between various metrics on Teichmüller spaces

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Abstract There are several Teichmüller spaces associated to a surface of infinite topological type, after the choice of a particular basepoint (a complex or a hyperbolic structure on the surface). Such spaces include the quasiconformal Teichmüller space, the length spectrum Teichmüller space, the Fenchel-Nielsen Teichmüller space, and there are others. In general, these spaces are set-theoretically different. An important question is therefore to understand relations between them. Each of these spaces is equipped with its own metric, and under some hypotheses, there are inclusions between them. In this paper, we obtain local metric comparison results on these inclusions, namely, we show that the inclusions are locally bi-Lipschitz under certain hypotheses. To obtain these results, we use some hyperbolic geometry estimates that give new results also for surfaces of finite type. We recall that in the case of a surface of finite type, all these Teichmüller spaces coincide setwise. In the case of a surface of finite type with no boundary components (but possibly with punctures), we show that the restriction of the identity map to any thick part of Teichmüller space is globally bi-Lipschitz with respect to the length spectrum metric on the domain and the classical Teichmüller metric on the range. In the case of a surface of finite type with punctures and boundary components, there is a metric on the Teichmüller space which we call the arc metric, whose definition is

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analogous to the length spectrum metric, but which uses lengths of geodesic arcs instead of lengths of closed geodesics. We show that the restriction of the identity map to any "relative thick" part of Teichmüller space is globally bi-Lipschitz, with respect to any of the three metrics: the length spectrum metric, the Teichmüller metric and the arc metric on the domain and on the range.

Keywords Teichmüller space · Teichmüller metric · Quasiconformal metric · Length spectrum metric · Fenchel-Nielsen coordinates · Fenchel-Nielsen metric

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1 Introduction

This paper concerns surfaces of finite and of infinite topological type. The results are different in each case, and we treat them separately. We start with the case of surfaces of infinite type.

Let Σ be a surface of infinite topological type, that is, a surface obtained by gluing a countably infinite number of generalized pairs of pants along their boundary components. Here, a generalized pair of pants is a sphere with three holes, a hole being either a point removed (giving rise to a puncture) or an open disk removed (giving rise to a boundary component). The hyperbolic structures we consider on Σ are such that the boundary components are closed geodesics and the punctures are cusps, that is, punctures admit neighborhoods that are quotients of regions of the form $\{(x, y) | y > a\}$ of the upper-half plane model of the hyperbolic plane by an isometry of the form $z \mapsto z + 1$.

There are several Teichmüller spaces associated with such a surface Σ, with several inclusions between them, and different metrics on them. We are interested in comparing these metrics, in cases where a comparison can be done. This paper is a continuation of the work done in [2] and [3], in which we introduced a space we called the Fenchel-Nielsen Teichmüller space, which is equipped with a metric we called the Fenchel-Nielsen metric. In these papers, we compared this metric with the Teichmüller metric. The definition of the Fenchel-Nielsen Teichmüller space of Σ depends on the choice of a base hyperbolic surface (considered as a basepoint for Teichmüller space) and of a pair of pants decomposition of that surface. Our work is also in the spirit of [12], in which we studied the various metrics on Teichmüller spaces of surfaces of infinite topological type. In the present paper, we mainly consider the question of local metric comparison between the Fenchel-Nielsen metric, the quasiconformal metric and the length spectrum metric.

Teichmüller spaces can be seen as parameter spaces for conformal structures on Σ. We will always consider these conformal structures as endowed with their *intrinsic metric*. This is a hyperbolic metric in the given conformal class, and it was defined by Bers. In the case of a Riemann surface with empty ideal boundary but which may have punctures (that is, ends with neighborhoods conformal to punctured discs), the intrinsic metric coincides with the Poincaré metric. But in the case of a Riemann surface with nonempty ideal boundary the two metrics do not coincide, see the end of Sect. 4 of [2] for the definition and a discussion. Endowing a Riemann surface with its intrinsic metric will allow us to use techniques from hyperbolic geometry, like the existence of geodesic pair of pants decompositions and of Fenchel-Nielsen coordinates. A geodesic pair of pants decomposition of a hyperbolic surface is a decomposition into pairs of pants, that is, spheres with three holes, by curves that are closed geodesics on the surface, and where each pair of pants in the decomposition can have 1, 2 or 3 geodesic boundary components, the other holes being cusps.

We note that in order to use Fenchel-Nielsen coordinates on a surface Σ, we need to show that given a topological pair of pants decomposition $\mathcal{P} = \{C_i\}_{i=1,2,\ldots}$ of Σ and a hyperbolic metric on Σ, there exists a unique geodesic pair of pants decomposition in which all the closed curves are homotopic to those in $\mathcal{P}$. This is not true for general hyperbolic metrics. One problem is that union of the geodesics obtained by replacing each curve C_i of $\mathcal{P}$ by its geodesic representative might not be closed, and there are other problems. In [2] we discussed this question and we gave a necessary and sufficient condition on hyperbolic structures on surfaces of infinite type to have a geodesic pair of pants decomposition. We called this condition Nielsen-convexity. One way of stating that result is to say that a hyperbolic metric satisfies this property if and only if it is the intrinsic metric of some conformal structure. This is the reason why in what follows we shall consider only hyperbolic metrics that are intrinsic in the sense we defined. We note that we proved in [2] that a hyperbolic surface is Nielsen-convex if and only if it is a convex core. An intrinsic metric is a convex core. This was proved by Bers, who showed that every Riemann surface is the Nielsen kernel of some Riemann surface (that he calls the Nielsen extension). To prove the existence of the Nielsen extension, Bers used the intrinsic metric on the surface, he showed that it is a convex core, and he then obtained the Nielsen extension by gluing to this convex core funnels and half-planes. The only fact that is useful to us here is that the intrinsic metric is Nielsen convex.

A hyperbolic metric S on Σ has *Fenchel-Nielsen coordinates* $((l_S(C_i), \theta_S(C_i)))_{i=1,2,\ldots}$ with respect to $\mathcal{P}$, using the notation of [2]. For the convenience of the reader, the definition of these coordinates as well as the precise definitions of the three Teichmüller spaces that we mentioned above are recalled in Sect. 2 below.

We consider a conformal structure S_0 on Σ which we consider as the basepoint of Teichmüller space. We denote by $(\mathcal{T}_{qc}(S_0), d_{qc})$ the quasiconformal Teichmüller space equipped with the corresponding metric, and by $(\mathcal{T}_{ls}(S_0), d_{ls})$ the length-spectrum Teichmüller space equipped with its metric. We also let $\mathcal{P} = \{C_i\}_{i=1,2,\ldots}$ be a fixed pair of pants decomposition of Σ and we denote by $(\mathcal{T}_{FN}(S_0), d_{FN})$ the resulting Fenchel-Nielsen Teichmüller space equipped with its metric. The Fenchel-Nielsen Teichmüller space depends on the choice of $\mathcal{P}$, but we will not mark this dependence explicitly unless this is necessary. Hence the space $\mathcal{T}_{FN}(S_0)$ and its metric are not intrinsic objects associated to S_0 but they constitute a useful tool to study the other spaces, because $\mathcal{T}_{FN}(S_0)$ has explicit coordinates and it is isometric to the sequence space ℓ^∞. We shall recall the definitions in Sect. 2.

We note that in this paper we consider the *reduced* Teichmüller space theory. This means that if the ideal boundary of S_0 is non-empty (see e.g. [16] for the definition of the ideal boundary), a Teichmüller space of Σ is a set of equivalence classes of marked Riemann surfaces up to homotopy, with the homotopy being free on the boundary components.

Given a hyperbolic structure S and a simple closed curve C on Σ, we denote by $l_S(C)$ the length of the unique S-geodesic in the homotopy class of C. In the case where S is a conformal structure on Σ, then we denote by $l_S(C)$ the length of the unique geodesic in the homotopy class of C with respect to the intrinsic metric associated to S.

We say that a conformal structure S is *upper-bounded with respect to* $\mathcal{P}$ if there exists a constant $M > 0$ such that for any simple closed curve C_i in $\mathcal{P}$, we have $l_S(C_i) \leq M$.

We say that a conformal structure is *upper-bounded* if it is upper-bounded with respect to some pair of pants decomposition, or if it is upper-bounded with respect to a pair of pants decomposition $\mathcal{P}$ which is understood.

A *marked conformal structure* (respectively a *marked hyperbolic structure*) on Σ is a pair (f, S) where S is a surface homeomorphic to Σ equipped with a conformal (respectively a hyperbolic structure) and $f : \Sigma \to S$ a homeomorphism. A marked conformal (respectively hyperbolic) structure on S induces a conformal (respectively hyperbolic) structure on the surface Σ itself by pull-back. Conversely, a conformal (respectively hyperbolic) structure on S can be considered as a marked hyperbolic surface, by taking the marking to be the identity homeomorphism of Σ. Using this formalism, an element of Teichmüller space is then an equivalence class of marked conformal (respectively hyperbolic) structures (f, S) where the equivalence relation $\sim$ is defined by $(f, S) \sim (f', S')$ if there exists a conformal homeomorphism (respectively an isometry) $h : S \to S'$ such that $h \circ f$ is homotopic to f'. We shall use the notation $[f, S]$ to denote the equivalence class of the marked surface (f, S).

In [2, Theorem 8.10], we proved the following:

Theorem 1.1 *Let S_0 be a conformal structure on Σ, and suppose that S_0 is upper-bounded. Then we have a set-theoretic equality $\mathcal{T}_{qc}(S_0) = \mathcal{T}_{FN}(S_0)$. Furthermore, the identity map*

$$j : (\mathcal{T}_{qc}(S_0), d_{qc}) \ni [f, S] \mapsto ((l_S(C_i), \theta_S(C_i)))_{i=1,2,\ldots} \in (\mathcal{T}_{FN}(S_0), d_{FN})$$

is a locally bi-Lipschitz homeomorphism.

Since the metric d_{FN} on the Fenchel-Nielsen Teichmüller space $\mathcal{T}_{FN}(S_0)$ makes this space isometric to the sequence space ℓ^∞, Theorem 1.1 gives a locally bi-Lipschitz homeomorphism between the quasiconformal Teichmüller space $(\mathcal{T}_{qc}(S_0), d_{ls})$ and ℓ^∞. An analogous result was proved by Fletcher in [8], in the setting of non-reduced Teichmüller spaces.

One of our goals in this paper is to give a local comparison result between the Fenchel-Nielsen metric and the length spectrum metric. The latter metric, in the setting of surfaces of infinite type, was first studied by Shiga in [17]. A famous lemma due to Wolpert (see the exposition in [1]) implies that for any hyperbolic surface S_0, we have a natural inclusion

$$\mathcal{T}_{qc}(S_0) \hookrightarrow \mathcal{T}_{ls}(S_0) \tag{1}$$

given by the identity map, and that this map is 1-Lipschitz, that is, for any two elements S and S' in $\mathcal{T}_{qc}(S_0)$, we have $d_{ls}(S, S') \leq d_{qc}(S, S')$. We note by the way that in general, this inclusion map is not surjective (see [12] for an example).

Theorem 1.1, combined with Wolpert's result, gives the following:

Theorem 1.2 *Let S_0 be a conformal structure on Σ which is upper-bounded. Then, for any S in $\mathcal{T}_{FN}(S_0)$, there exists a neighborhood N of S in $\mathcal{T}_{FN}(S_0)$ and a constant $C > 0$ that depends only on N such that for any S' and S'' in N, we have*

$$d_{ls}(S', S'') \leq C d_{FN}(S', S'').$$

Besides the upper-boundedness property for conformal structures, we shall use the following stronger property, which we call *Shiga's property*, because it was used in a similar context in Shiga's paper [17].

We say that a conformal structure S *satisfies Shiga's property with respect to* $\mathcal{P}$ if there exist two positive constants δ and M such that the following holds

$$\forall C_i \in \mathcal{P},\ \delta \leq l_S(C_i) \leq M. \tag{2}$$

Like for the upper-boundedness condition, we shall say that a conformal structure *satisfies Shiga's property* if it satisfies such a property for some pair of pants decomposition, or if it satisfies it for a pair of pants decomposition which is understood.

One of the main results in this paper is the following, the proof of which appears at the end of Sect. 3 (Theorem 3.5).

Theorem 1.3 *Let S_0 be a conformal structure on Σ satisfying Shiga's condition (2) and let $\mathcal{T}(S_0)_{qc}$ be the corresponding quasiconformal Teichmüller space. For any element S of $\mathcal{T}_{qc}(S_0)$ and for any positive constant D, there exists a positive real number C that depends only on δ, M, D and $d_{ls}(S_0, S)$ such that if two elements S_1 and S_2 of $\mathcal{T}_{qc}(S_0)$ are in the open ball of centre S and radius D, then $d_{FN}(S_1, S_2) < C d_{ls}(S_1, S_2)$.*

From Theorems 1.1, 1.2 and 1.3, we deduce the following.

Theorem 1.4 *Let S_0 be a conformal structure on Σ satisfying Shiga's condition. Then we have a set-theoretic equality $\mathcal{T}_{qc}(S_0) = \mathcal{T}_{ls}(S_0) = \mathcal{T}_{FN}(S_0)$, and the identity map between any two of the three spaces with their respective metrics d_{qc}, d_{ls} and d_{FN} is locally bi-Lipschitz.*

This implies in particular that under Shiga's condition, $\mathcal{T}_{ls}(S_0)$ is locally bi-Lipschitz equivalent to the sequence space ℓ^∞. It also implies that the Fenchel-Nielsen Teichmüller space $\mathcal{T}_{FN}(S_0)$, as a set, does not depend on the choice of the pair of pants decomposition of S_0, and that the identity map between two Fenchel-Nielsen spaces with the same basepoint and corresponding to different pairs of pants decompositions is a locally bi-Lipschitz homeomorphism. (In particular, the topologies induced are the same.)

In particular, under Shiga's condition, $\mathcal{T}_{ls}$ is, like the other two spaces, contractible.

We then show (Theorem 4.5 in Sect. 4) the following result which shows that if we remove the hypothesis on Shiga's condition in Theorem 1.3, the conclusion may not hold:

Theorem 1.5 *If S_0 is a conformal surface of infinite topological type with a pair of pants decomposition $\mathcal{P} = \{C_i\}$ such that there is a subsequence of $\{C_{i_k}\}$ contained in the interior of S_0 whose hyperbolic lengths tend to zero, then the identity map between the Teichmüller space $(\mathcal{T}_{qc}(S_0), d_{qc})$ and its image in $(\mathcal{T}_{ls}(S_0), d_{qc})$ is not locally bi-Lipschitz.*

(Recall that by Wolpert's inequality there is always a set-theoretic inclusion $\mathcal{T}_{qc}(S_0) \subset \mathcal{T}_{ls}(S_0)$.)

The above results and their proofs, although they are formulated for surfaces of infinite topological type, apply with little changes to surfaces of finite topological type. In the latter case, all Teichmüller spaces coincide setwise. Some of the results we obtain here for surfaces of infinite type are known to be true for surfaces of finite type, but we also obtain some new results. We consider the case of surfaces of finite topological type in Sect. 5.

We first have the followig:

Corollary 1.6 *For any Riemann surface of finite topological type and of negative Euler characteristic which is not homeomorphic to a pair of pants, the identity map between the Teichmüller and the length spectrum metrics on $\mathcal{T}(S)$ is not a quasi-isometry.*

In the case of a surface of finite type with punctures and nonempty boundary, we introduced in the papers [13] and [14] a metric on Teichmüller space which we called the *arc metric* and which we denoted by δ_L. The definition is analogous to the length spectrum metric, but it uses lengths of arcs instead of lengths of closed curves. This definition is recalled below (Eq. (13). (Note that in the notation δ_L, L is just part of the notation, used to recall the word "Length"; it is not a parameter.)

For such a surface, we let $\mathcal{D}$ be the set of boundary components of S.

We recall that for a surface of finite type and for $\epsilon > 0$, the ϵ-thick part of Teichmüller space, denoted by $\mathcal{T}_\epsilon$, is defined as the space

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \forall \gamma \in \mathcal{S}, l_X(\gamma) \geq \epsilon\}$$

where $\mathcal{S}$ denotes the set of homotopy classes of simple closed curves on Σ that are not homotopic to a point or to a puncture (but they can be homotopic to a boundary component).

For surfaces of finite type with nonempty boundary, for $\epsilon > 0$ and $\epsilon_0 \geq \epsilon$, we introduce the ϵ_0-*relative* ϵ-thick part of Teichmüller space, denoted by $\mathcal{T}_{\epsilon,\epsilon_0}$, as the subset of the ϵ-thick part of Teichmüller space defined as

$$\mathcal{T}_{\epsilon,\epsilon_0}(S) = \{X \in \mathcal{T}(S) \mid \forall\gamma \in \mathcal{S}, l_X(\gamma) \geq \epsilon \text{ and } \forall\delta \in \mathcal{D}, l_X(\gamma) \leq \epsilon_0\}.$$

We prove the following (Theorem 5.3 below):

Theorem 1.7 *Let S be a topologically finite type surface. For any $\epsilon > 0$ and any $\epsilon_0 \geq \epsilon$, the identity map between any two of the three metrics d_{ls}, d_{qc} and δ_L on $\mathcal{T}_{\epsilon,\epsilon_0}(S)$ is globally bi-Lipschitz, with a bi-Lipschitz constant depending only on the topological type of S, on ϵ and on ϵ_0.*

In the case where the surface S is of finite type and with empty boundary, then we also have a similar statement for the ϵ-thick part of Teichmüller space (Theorem 5.4 below).

2 The three Teichmüller spaces

In order to make the paper self-contained and for the convenience of the reader, we recall the precise definitions of the three Teichmüller spaces that we associate to a surface of infinite topological type, namely, the quasiconformal Teichmüller space, the Fenchel-Nielsen Teichmüller space and the length-spectrum Teichmüller space. These spaces were considered in the papers [2,3], and [12].

We start with the quasiconformal Teichmüller space $\mathcal{T}_{qc}(S_0)$. In this definition the hyperbolic metrics do not play a significant role, and when such a metric appears in the quasiconformal Teichmüller space, we only use its underlying Riemann surface structure. More precisely, the elements of $\mathcal{T}_{qc}(S_0)$ are the homotopy classes of conformal structures S on Σ such that the identity map between Σ equipped with S_0 and S on the domain and on the target respectively is homotopic to a quasiconformal map. The space $\mathcal{T}_{qc}(S_0)$ is equipped with the *quasiconformal metric*, also called the *Teichmüller metric*, in which for any two homotopy classes of conformal structures (Σ, S) and (Σ, S'), their *quasiconformal distance* $d_{qc}(S, S')$ is defined as

$$d_{qc}(S, S') = \frac{1}{2}\log\inf\{K(f)\} \tag{3}$$

where the infimum is taken over the set of quasiconformal dilatations $K(f)$ of quasiconformal homeomorphisms $f : (\Sigma, S) \to (\Sigma, S')$ which are homotopic to the identity. Here, we are using the notation (Σ, S) to say that S is a marked structure (conformal or hyperbolic) on the surface S, with the marking being the identity map.

The conformal structure S_0 is the *basepoint* of $\mathcal{T}_{qc}(S_0)$.

We now recall the definition of the Fenchel-Nielsen Teichmüller spaces $\mathcal{T}_{FN}(S_0)$. In this definition we use the intrinsic hyperbolic metric associated to a conformal structure, and we refer the reader to the discussion in the introduction regarding the pair of pants decomposition rendered geodesic with respect to the intrinsic hyperbolic metric. The definition of $\mathcal{T}_{FN}(S_0)$ is relative to the choice of a (topological) pair of pants decomposition $\mathcal{P} = \{C_i\}$ of Σ, and to the Fenchel-Nielsen coordinates associated to that decomposition. The definition of the Fenchel-Nielsen parameters is similar to the one that is done in the case of surfaces of finite type, and we considered them in detail for surfaces of infinite type in [2].

Let S be a (homotopy class of a conformal) structure on Σ. To each homotopy class of closed geodesics $C_i \in \mathcal{P}$, we consider its *length parameter* $l_S(C_i)$ as defined in §1 above, and its *twist parameter* $\theta_S(C_i)$, which is defined only if C_i is not the homotopy class of a boundary component of Σ. The twist parameter is a measure of the relative twist amount along the geodesic in the class C_i between the two generalized pairs of pants that have this geodesic in common. The twist amount per unit time along the (geodesic in the class) C_i is chosen so that a complete positive Dehn twist along C_i changes the twist parameter by addition of 2π.

Thus, for any conformal structure on S, its *Fenchel-Nielsen parameters* relative to $\mathcal{P}$ is the collection of pairs

$$((l_S(C_i), \theta_S(C_i)))_{i=1,2,\dots}$$

where it is understood that if C_i is homotopic to a boundary component, then there is no twist parameter associated to it, and instead of a pair $(l_S(C_i), \theta_S(C_i))$, we have a single parameter $l_S(C_i)$.

Now given two conformal structures S and S' on Σ, their *Fenchel-Nielsen distance* (with respect to $\mathcal{P}$) is

$$d_{FN}(S, S') = \sup_{i=1,2,\dots} \max \left(\left| \log \frac{l_S(C_i)}{l_{S'}(C_i)} \right|, |l_S(C_i)\theta_S(C_i) - l_{S'}(C_i)\theta_{S'}(C_i)| \right), \tag{4}$$

again with the convention that if C_i is the homotopy class of a boundary component of Σ, then there is no twist parameter to be considered.

Two conformal structures S and S' on Σ are said to be *Fenchel-Nielsen bounded* (relatively to $\mathcal{P}$) if their Fenchel-Nielsen distance is finite. Fenchel-Nielsen boundedness is an equivalence relation.

We say that two hyperbolic structures S and S' on Σ are equivalent if there exists an isometry $(\Sigma, S) \to (\Sigma, S')$ which is homotopic to the identity. Now given our basepoint S_0 of Teichmüller space, the *Fenchel-Nielsen Teichmüller space* with respect to $\mathcal{P}$ and with basepoint S_0, denoted by $\mathcal{T}_{FN}(S_0)$, is the space of equivalence classes of conformal structures that are Fenchel-Nielsen bounded from S_0 relative to $\mathcal{P}$.

The function d_{FN} defined above is a distance function on $\mathcal{T}_{FN}(S_0)$ and we call it the *Fenchel-Nielsen distance* relative to the pair of pants decomposition $\mathcal{P}$. The map

$$\mathcal{T}_{FN}(S_0) \ni H \mapsto (\log(l_H(C_i)) - \log(l_{S_0}(C_i)), l_H(C_i)\theta_H(C_i))_{i=1,2,\dots} \in \ell^\infty$$

is an isometric bijection between $\mathcal{T}_{FN}(S_0)$ and the sequence space ℓ^∞. It follows from known properties of ℓ^∞-norms that the Fenchel-Nielsen distance on $\mathcal{T}_{FN}(S_0)$ is complete.

Finally, we recall the definition of the length-spectrum Teichmüller space $\mathcal{T}_{ls}(S_0)$ with basepoint S_0. Again, in this definition we use the intrinsic hyperbolic metric associated to a conformal structure, see the discussion in the introduction.

We recall that $\mathcal{S}$ denotes the set of homotopy classes of simple closed curves on Σ that are not homotopic to a point or to a puncture. We first define the *length-spectrum constant* $L(f)$ of a homeomorphism $f : (\Sigma, S) \to (\Sigma, S')$ where S and S' are two (homotopy classes of) conformal structures on Σ as

$$L(f) = \sup_{\alpha \in \mathcal{S}} \left\{ \frac{l_{S'}(f(\alpha))}{l_S(\alpha)}, \frac{l_S(\alpha)}{l_{S'}(f(\alpha))} \right\}.$$

This quantity $L(f)$ depends only on the homotopy class of f, and we say that f is *length-spectrum bounded* if $L(f) < \infty$.

We consider that two hyperbolic metrics (Σ, S) and (Σ, S') on Σ are equivalent if there exists an isometry (or, equivalently, a length spectrum preserving homeomorphism) from (Σ, S) to (Σ, S') which is homotopic to the identity. The *length spectrum Teichmüller space* $\mathcal{T}_{ls}(S_0)$ of Σ with basepoint S_0 is the set of homotopy classes of conformal structures S on Σ such that the identity map Id : $(\Sigma, S_0) \to (\Sigma, S)$ is length-spectrum bounded.

The *length-spectrum* metric d_{ls} on $\mathcal{T}_{ls}(S_0)$ is defined by taking the distance $d_{ls}(S, S')$ between two points in that space to be

$$d_{ls}(S, S') = \frac{1}{2} \log L(\mathrm{Id}). \tag{5}$$

where Id is the identity map between (Σ, S) and (Σ, S'). (We note that the length-spectrum constant of a length-spectrum bounded homeomorphism depends only on the homotopy class of such a homeomorphism.)

3 On the Fenchel-Nielsen distance and the length spectrum distance

Let S be a hyperbolic structure on the surface of infinite topological type S and let $\mathcal{P} = \{C_i\}$ be a geodesic pair of pants decomposition of S.

Lemma 3.1 *Let $\delta < M$ be two positive constants such that each $C_i \in \mathcal{P}$ satisfies $\delta \leq l_S(C_i) \leq M$. Then, for each $C_i \in \mathcal{P}$, we can find a simple closed geodesic β_i satisfying the following properties:*

(1) *β_i intersects C_i in a minimal number of points (this number is one or two);*
(2) *β_i does not intersect C_j, for any $j \neq i$;*
(3) *there is a constant L depending only on δ and M such that $l_S(\beta_i) < L$;*
(4) *the sine of the intersection angle (or of the two angles) of β_i with C_i is bounded from below by a positive constant that depends only on M.*

Proof Topologically, the curves β_i are represented in Fig. 1. Using the inequalities $\delta \leq l_S(C_i) \leq M$, an upper bound L for $l_S(\beta_i)$ is obtained by estimates on hyperbolic right-angled hexagons and pentagons. Using the upper bound L for $l_S(\beta_i)$ and the upper bound M for $l_S(C_i)$, we can prove that $\sin\theta$ has a positive lower bound depending on L and M. We refer to Lemma 7.5 in [2] for the details of the proof. □

Now we fix an element $C_i \in \mathcal{P}$ and we let $\tau^t : S \to S_t$ be the time-t Fenchel-Nielsen left twist deformation of S along C_i. (At time t, we twist by an amount equal to t measured on the curve C_i.)

Let β be a simple closed geodesic on S. For all t in $\mathbb{R}$, we denote by β_t the simple closed geodesic in S_t homotopic to $\tau^t(\beta)$ and we let $l_t(\beta) = l_{S_t}(\beta_t)$ be its hyperbolic length. (Note that the class of β_t is the same as the class of β when we consider the hyperbolic structures as being on the same fixed base surface, or as marked surfaces with respect to a fixed base surface). The intersection angle of C_i and β_t at a point $p \in C_i \cap \beta_t$ (measured from C_i to β_t) is denoted by $\theta_t(p)$.

All angles used in this paper take their values in the interval $[0, \pi]$.

We shall use the following formulae due to Wolpert [19], concerning the first and second derivatives of the Fenchel-Nielsen flow. We use the formulation in Weiss [18] p. 281).

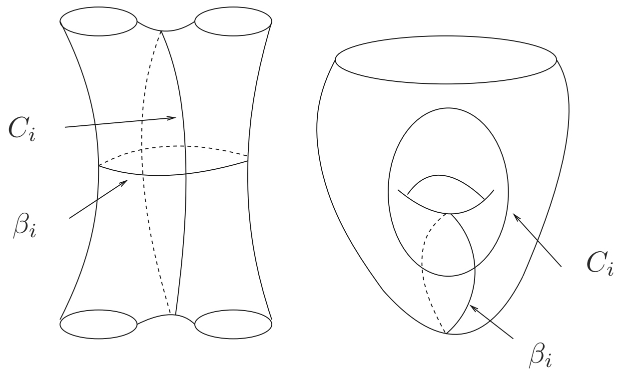

Fig. 1 The two cases for the curve β_i used in the proof of Lemma 3.1. In each case, we have represented the simple *closed curves* C_i and β_i

Lemma 3.2 *For any simple closed geodesic* β*, the function* $t \mapsto l_t(\beta)$ *is real-analytic, and we have*

$$\frac{dl_t(\beta)}{dt}|_{t=0} = \sum_{p\in C_i\cap\beta} \cos\theta(p)$$

and

$$\frac{d^2l_t(\beta)}{dt^2}|_{t=0} = \sum_{p,q} \frac{e^{l_1}+e^{l_2}}{2(e^{l(\beta)}-1)} \sin\theta(p)\sin\theta(q) + \sum_{p} \frac{e^{l(\beta)}+1}{2(e^{l(\beta)}-1)} \sin^2\theta(p).$$

In the right hand side of the last equality, the first sum is taken over the set of ordered pairs of distinct points p, q *in* $C_i \cap \beta$*, and* l_1 *and* l_2 *are the lengths of the two subarcs that they subdivide on* β*, and the second sum is taken over all points* p *in* $C_i \cap \beta$*.*

We shall use special cases of the above formulae, where β intersects C_i either in one or in two points.

In the case where β and C_i have only one intersection point p, with intersection angle $\theta(p)$, the formulae become

$$\frac{dl_t(\beta)}{dt}|_{t=0} = \cos\theta(p)$$

and

$$\frac{d^2l_t(\beta)}{dt^2}|_{t=0} = \frac{e^{l(\beta)}+1}{2(e^{l(\beta)}-1)} \sin^2\theta(p).$$

In the case where β and C_i have exactly two intersection points, denoted by p_1 and p_2, the formulae become

$$\frac{dl_t(\beta)}{dt}|_{t=0} = \cos\theta(p_1) + \cos\theta(p_2)$$

and

$$\frac{d^2l_t(\beta)}{dt^2}|_{t=0} = \frac{e^{l_1}+e^{l_2}}{(e^{l(\beta)}-1)} \sin\theta(p_1)\sin\theta(p_2) + \frac{e^{l(\beta)}+1}{2(e^{l(\beta)}-1)}(\sin^2\theta(p_1)+\sin^2\theta(p_2)).$$

We now consider multi-twists, that is, compositions of twist maps along a family of disjoint simple closed curves.

We let $t = (t_i)$, $i = 1, 2, \dots$ be a sequence of real numbers, and for any real number t we denote by $\tau^t : S \to S_t$ the multi-twist obtained by twisting an amount of t_i along each curve C_i.

For $t = (t_i)$, $i = 1, 2, \dots$, we set $|t| = \sup_{i=1,2,\dots} |t_i|$.

Proposition 3.3 *Assume there exist two positive constants δ and M such that each $C_i \in \mathcal{P}$ satisfies $\delta \le l_S(C_i) \le M$. If $\sup_{\beta \in \mathcal{S}} |\log \frac{l_t(\beta)}{l(\beta)}| \le D$, then*

$$|t| < C \sup_{\beta \in \mathcal{S}} \left| \log \frac{l_t(\beta)}{l(\beta)} \right|,$$

where C is a constant depending on δ, M and D.

Proof It suffices to prove that for all $i = 1, 2, \dots$, $|t_i| \le C \sup_{\beta \in S} |\log \frac{l_t(\beta)}{l(\beta)}|$, where C is a constant that depends on δ, M and D and that does not depend on i.

For each i, we let β_i be the simple closed geodesic given by Lemma 3.1. We shall apply the hypothesis $|\log \frac{l_t(\beta)}{l(\beta)}| \le D$ to $\beta = \beta_i$ and show that

$$\forall i = 1, 2, \dots, |t_i| \le C \left| \log \frac{l_t(\beta_i)}{l(\beta_i)} \right|. \tag{6}$$

Note that the length $l_t(\beta_i)$ is affected by the twist along C_i, and not by any twist along β_j for $j \ne i$.

From (6), we will then get

$$|t| = \sup\{|t_i|\} \le C \sup_{\beta_i} \left| \log \frac{l_t(\beta_i)}{l(\beta_i)} \right| \le \sup_{\beta \in \mathcal{S}} \left| \log \frac{l_t(\beta)}{l(\beta)} \right|,$$

which is what we need to prove.

Thus, we now prove (6). We only need to assume that

$$\sup_{\beta_i} \left| \log \frac{l_t(\beta_i)}{l(\beta_i)} \right| \le D,$$

which is weaker than our assumption that

$$\sup_{\beta \in \mathcal{S}} \left| \log \frac{l_t(\beta)}{l(\beta)} \right| \le D.$$

Without loss of generality, we can assume that $t_i > 0$. In the following estimates, we can restrict our attention to the pair of pants (or two pairs of pants) that contains C_i, since we only need to consider the ratio $\frac{l_t(\beta)}{l(\beta)}$. We denote $t_i = t$ for simplicity.

There are two cases:

Case 1 : C_i intersects β_i at a single point $p \in S$. Let θ be the angle at that intersection point.

By Lemma 3.1, there are positive constants $\rho_0 = \rho_0(M)$ and $L = L(\delta, M)$ such that $\sin\theta \ge \rho_0$, $l(\beta_i) < L$. Since the function $t \mapsto l_t(\beta)$ is real-analytic, we can write down the second-order Taylor expansion at each t

$$l_{t'}(\beta) = l_t(\beta) + \frac{dl_t(\beta)}{dt}(t' - t) + \frac{d^2 l_t(\beta)}{dt^2}\frac{(t' - t)^2}{2} + o(|t' - t|^2).$$

From Lemma 3.2, we obtain

$$l_{t'}(\beta_i) = l_t(\beta_i) + \cos\theta_t \cdot (t' - t) + \frac{e^{l_t(\beta_i)} + 1}{4(e^{l_t(\beta_i)} - 1)} \sin^2\theta_t \cdot (t - t')^2 + o(t^2), \tag{7}$$

where θ_t, is, as above, the intersection angle of C_i and β in the hyperbolic metric S_t.

We now use the following result of Kerckhoff [9]:

Lemma 3.4 *The function $t \mapsto l_t(\beta_i)$ is strictly convex and the function $t \mapsto \cos\theta_t$ is strictly increasing.*

In particular, if $l_t(\beta_i)$ attains its minimum at t_0, then $\cos\theta_{t_0} = 0$ (or, equivalently, $\theta_{t_0} = \frac{\pi}{2}$). When $t < t_0$, $\cos\theta_t < 0$ and when $t > t_0$, $\cos\theta_t > 0$.

We set $|\log\frac{l_t(\beta_i)}{l(\beta_i)}| = \eta \leq D$. Then $e^{-\eta} \leq \frac{l_t(\beta_i)}{l(\beta_i)} \leq e^{\eta}$. Since $1 - e^{\eta} \leq e^{-\eta} - 1 \leq \frac{l_t(\beta_i)}{l(\beta_i)} - 1 \leq e^{\eta} - 1$, we have

$$\begin{aligned} |l_t(\beta_i) - l(\beta_i)| &= l(\beta_i)\left|\frac{l_t(\beta_i)}{l(\beta_i)} - 1\right| \\ &\leq l(\beta_i)|e^{\eta} - 1| \\ &= l(\beta_i)\left(\eta + \sum_{n\geq 2}\frac{\eta^n}{n!}\right). \end{aligned}$$

By assumption, $l(\beta_i) < L$ and $\eta \leq D$. As a result,

$$|l_t(\beta_i) - l(\beta_i)| \leq L\left(1 + \sum_{n\geq 2}\frac{D^{n-1}}{n!}\right)\eta = e(D)L\eta, \tag{8}$$

where $e(D)$ is a constant that depends only on D.

Let $\lambda > 0$ be a fixed sufficiently small constant, to be determined later.

First assume that $\cos\theta \geq \lambda$. Applying the mean value theorem to the function $f(t) = l_t(\beta_i)$ on the interval $[0, t]$ and using the fact that $f'(t) = \cos\theta_t$, we have

$$l_t(\beta_i) - l(\beta_i) = \cos\theta_\xi t, \tag{9}$$

for some $\xi \in [0, t]$.

Since $\cos\theta_\xi \geq \cos\theta \geq \lambda$ (Lemma 3.4), combining (9) with (8), we have

$$t = \frac{l_t(\beta_i) - l(\beta_i)}{\cos\theta_\xi} \leq \frac{e(D)L\eta}{\lambda}.$$

If $|\cos\theta| < \lambda$, we let β_i' be the unique geodesic on S homotopic to the image of β_i under the action of a positive Dehn twist along C_i. Note that the hyperbolic length of β_i' is bounded by $L + M$ and, in fact, $\beta_i' = l_T(\beta_i)$, where $T = l_S(C_i) \geq \delta$. The value T is the time needed for a full Dehn twist along β_i.

It is clear that β_i' also satisfies the properties (1)–(3) in Lemma 3.1. Property (4) follows then from these three (see the proof of Lemma 7.5 in [2]). Let p' be the intersection point of C_i with β_i' and θ' be the corresponding intersection angle. Thus, there is a positive constant $\rho_1 = \rho_1(L + M)$ such that $\sin\theta' \geq \rho_1$. Let $\rho = \min\{\rho_0, \rho_1\}$.

We want to give a positive lower bound for $\cos\theta'$.

Since the hyperbolic length of $l_t(\beta_i), 0 \le t \le T$ is bounded above by $L + M$ and since $\frac{e^x+1}{e^x-1}$ is a strictly decreasing function of x, we have

$$\frac{e^{l_t(\beta_i)}+1}{e^{l_t(\beta_i)}-1} > \frac{e^{L+M}+1}{e^{L+M}-1}, \text{ for } 0 \le t \le T. \tag{10}$$

Let us set $K = \frac{e^{L+M}+1}{4(e^{L+M}-1)}$.

From Wolpert's formula (Lemma 3.2), the second derivative with respect to t of the length function $l_t(\beta_i)$ is equal to

$$\frac{e^{l_t(\beta_i)}+1}{2(e^{l_t(\beta_i)}-1)} \sin^2\theta_t. \tag{11}$$

Inequality (10) shows that

$$\forall t \in [0, T], \ \frac{e^{l_t(\beta_i)}+1}{2(e^{l_t(\beta_i)}-1)} \sin^2\theta_t > K \sin^2\theta_t. \tag{12}$$

Thus, for $0 \le t \le T$, the second derivative of $l_t(\beta_i)$ with respect to t is bounded below by $K \sin^2\theta_t$.

For $0 \le t \le T$, we have $\sin\theta_t \ge \min\{\sin\theta, \sin\theta_T = \sin\theta'\} \ge \rho$, since $\sin\theta \ge \rho_0$ and $\sin\theta' \ge \rho_1$.

Thus, we have, using (12),

$$\frac{d\cos\theta_t}{dt} = \frac{d^2 l_t(\beta_i)}{dt^2} \ge K\rho^2.$$

As a result, and applying again the mean value theorem,

$$\cos\theta' - \cos\theta \ge K\rho^2 T \ge K\rho^2\delta.$$

Now we set $\lambda = \frac{K\rho^2\delta}{2}$. Since $|\cos\theta| < \lambda$ and $\cos\theta' - \cos\theta \ge 2\lambda$, we have $\cos\theta' > \lambda$. The same arguments used in the first subcase show that

$$t \le \frac{e(D)(L+M)\eta}{\lambda}.$$

The remaining subcase is when $\cos\theta \le -\lambda$.

Since $\sin\theta \ge \rho_0$, we have $-\sqrt{1-\rho_0^2} \le \cos\theta < -\lambda$. By Lemma 3.4, if $l_t(\beta_i)$ attains its minimum at t_0, then $t_0 > 0$. This uses the fact that $\cos\theta < 0$. Set $N = [t_0] + 1$. Since $l_t(\beta_i)$ decreases and $\sin\theta_t$ increases when $t \in [0, t_0]$, we have

$$\frac{e^{l_t(\beta_i)}+1}{e^{l_t(\beta_i)}-1} \sin\theta_t^2 \ge \frac{e^{l(\beta_i)}+1}{e^{l(\beta_i)}-1} \sin\theta^2 \ge \frac{e^L+1}{e^L-1}\rho_0^2, \text{ for } t \in [0, t_0].$$

As a result, the first order derivative of $\cos\theta_t$ satisfies

$$\frac{d\cos\theta_t}{dt} \ge \frac{e^L+1}{4(e^L-1)}\rho_0^2, \text{ for } t \in [0, t_0].$$

Note that t_0 is exactly the value when $\cos\theta_t$ equals 0, and so it follows that

$$\sqrt{1-\rho_0^2} \ge \cos\theta_{t_0} - \cos\theta \ge \frac{e^L+1}{4(e^L-1)}\rho_0^2 t_0.$$

This shows that N is bounded above by

$$\frac{4(e^L-1)}{(e^L+1)}\frac{\sqrt{1-\rho_0^2}}{\rho_0^2}+1.$$

Let β_i^N be the geodesic on S homotopic to the image of β_i under an N-order Dehn twist along C_i. The intersection angle θ_N of C_i and β_i^N satisfies $\cos\theta_N>0$ and the length of β_i^N is bounded above by $L+NM$. By repeating the same argument as above, we complete the proof of Case (**I**).

Case 2 : β_i intersects C_i at two different points p_1, p_2 (and we denote the intersecting angle by θ_1 and θ_2 respectively).

In this case, we consider the formula

$$l_t(\beta_i)=l(\beta_i)+(\cos\theta_1+\cos\theta_2)\cdot t + \left(\frac{e^{l_1}+e^{l_2}}{(e^{l(\beta_i)}-1)}\sin\theta_1\sin\theta_2+\frac{e^{l(\beta_i)}+1}{2(e^{l(\beta_i)}-1)}(\sin^2\theta_1+\sin^2\theta_2)\right)\cdot\frac{t^2}{2}+o(t^2).$$

By Lemma 3.1, there are positive constants $\rho_0=\rho_0(M)$ and $L=L(\delta,M)$ such that $\sin\theta_1,\sin\theta_2\geq\rho_0$, and $l(\beta_i)<L$ (and then $\frac{1}{e^{l(\beta_i)}-1}>\frac{1}{e^L-1}$). As a result, one checks that $\frac{e^{l_1}+e^{l_2}}{(e^{l(\beta_i)}-1)}\sin\theta_1\sin\theta_2+\frac{e^{l(\beta_i)}+1}{2(e^{l(\beta_i)}-1)}(\sin^2\theta_1+\sin^2\theta_2)\geq A$, for some constant A depending on ρ_0 and L.

Now fix a sufficiently small constant $0<\lambda_0<\frac{A}{2}\delta$. If $\cos\theta_1+\cos\theta_2\geq\lambda_0$, using again the mean value theorem for the function $t\mapsto l_t(\beta_i)$, it is easy to show that

$$|t|<\frac{e(D)L\eta}{\lambda_0}.$$

If $|\cos\theta_1+\cos\theta_2|<\lambda_0$, then we replace β_i by its image β_i' under the action of positive Dehn twist along C_i. Let θ_1' and θ_2' be the intersection angles of C_i and β_i', then the same proof as Case 1 shows that $|\cos\theta_1'+\cos\theta_2'|\geq\lambda_0$. As a result, we also have $|t|<\frac{e(D)(L+2M)}{\lambda_0}\eta$. If $\cos\theta_1+\cos\theta_2\leq-\lambda_0$, then we have to replace β_i by the image of β_i under the action of an $N-$order Dehn twist along C_i, denoted by β_i^N, such that $l(\beta_i^N)<L+NM$ and the two intersection angles of C_i and β_i^N are non-negative. To give a upper bound of N, we use the same argument as the last step of Case 1 , observing that the two intersection angles have the same behavior under a Fenchel-Nielsen twist deformation. □

Now we can prove the following

Theorem 3.5 *Let S_0 be a conformal structure satisfying Shiga's condition (2), let $\S(S_0)$ be the corresponding length-spectrum Teichmüller space and let S be a point in $\mathcal{T}_{ls}(S_0)$. If $d_{ls}(S,S_1)\leq D$ and $d_{ls}(S,S_2)\leq D$ for some positive real number D, then $d_{FN}(S_1,S_2)<Cd_{ls}(S_1,S_2)$, where C is a positive constant that depends only on δ, M, D.*

Proof There are positive constants δ_1 and M_1, depending on δ, M and D, such that for each $C_i\in\mathcal{P}$, its hyperbolic length in S_1 satisfies

$$\delta_1\leq l_{S_1}(C_i)\leq M_1.$$

By assumption, $d_{ls}(S_1, S_2) \leq 2D$. Then by Proposition 3.3,

$$\begin{aligned} d_{FN}(S_1, S_2) &\leq d_{FN}(S_1, S) + d_{FN}(S, S_2) \\ &< (C+1)d_{ls}(S_1, S_2) \end{aligned}$$

where C is a positive constant depending on δ_1, M_1 and D. □

4 The Teichmüller distance and the length spectrum distance

In this section, we show that the result in Theorem 1.4 is false if we remove Shiga's condition.

Let S_0 be a conformal structure on the surface (of infinite type) Σ and $\mathcal{T}_{qc}(S_0)$ be its quasiconformal Teichmüller space, let S be an element of $\mathcal{T}_{qc}(S_0)$ and let α be a simple closed geodesic on S. As before, we denote by S_t be the hyperbolic surface obtained by the time-t Fenchel-Nielsen twist deformation of S along α. Recall the following proposition, which is a direct corollary of Lemma 7.4 in [2].

Proposition 4.1 *Let T be a positive constant. For $|t| < T$, we have*

$$d_{qc}(S, S_t) \geq C|t|,$$

where C is a positive constant depending only on T.

To compare the Teichmüller distance $d_{qc}(S, S_t)$ and the length spectrum distance $d_{ls}(S, S_t)$, we show the following inequality.

Lemma 4.2 *For $|t| \leq |2 \log l_s(\alpha)|$, we have:*

$$d_{ls}(S, S_t) \leq \frac{1}{2} \log \sup_{\gamma | i(\alpha,\gamma) \neq 0} \frac{i(\alpha, \gamma)|t|}{l_S(\gamma)}.$$

Proof Without loss of generality, we can assume that $t > 0$. For any simple closed curve β satisfying $i(\beta, \alpha) \neq 0$, let $l_{S_t}(\beta)$ denote the hyperbolic length of β in S_t. From the definition of the Fenchel-Nielsen twist, we easily have

$$l_S(\beta) - i(\alpha, \beta)t \leq l_{S_t}(\beta) \leq l_S(\beta) + i(\alpha, \beta)t.$$

Note that by the collar lemma, left-hand side is positive for $|t| \leq |2 \log l_S(\alpha)|$.

The length spectrum distance can be written as

$$d_{ls}(S, S_t) = \max \left\{ \frac{1}{2} \log \sup_{\gamma} \frac{l_{S_t}(\gamma)}{l_S(\gamma)}, \frac{1}{2} \log \sup_{\gamma} \frac{l_S(\gamma)}{l_{S_t}(\gamma)} \right\},$$

where the supremum is taken over all essential simple closed curves γ.

The hyperbolic length of a (homotopy class of) simple closed curve γ satisfying $i(\alpha, \gamma) = 0$ is invariant under the twist along α. As a result, we have

$$d_{ls}(S, S_t) = \max \left\{ \frac{1}{2} \log \sup_{\gamma | i(\alpha,\gamma) \neq 0} \frac{l_{S_t}(\gamma)}{l_S(\gamma)}, \frac{1}{2} \log \sup_{\gamma | i(\alpha,\gamma) \neq 0} \frac{l_S(\gamma)}{l_{S_t}(\gamma)} \right\}.$$

For any simple closed curve γ with $i(\alpha, \gamma) \neq 0$,

$$\log \frac{l_{S_t}(\gamma)}{l_S(\gamma)} \leq \left| \log \frac{l_S(\gamma) + i(\alpha, \gamma)t}{l_S(\gamma)} \right| \leq \frac{i(\alpha, \gamma)t}{l_S(\gamma)}$$

and

$$\log \frac{l_S(\gamma)}{l_{S_t}(\gamma)} \leq |\log \frac{l_S(\gamma)}{l_S(\gamma) - i(\alpha,\gamma)t}| \leq \frac{i(\alpha,\gamma)t}{l_S(\gamma)}.$$

Then we have

$$d_{ls}(S, S_t) \leq \frac{1}{2} \sup_{\gamma | i(\alpha,\gamma)\neq 0} \frac{i(\alpha,\gamma)t}{l_S(\gamma)}.$$

□

Note that if $l_S(\alpha) \leq L$, then it follows from the collar lemma that there is a constant C depending on L such that for any simple closed geodesic γ with $i(\alpha,\gamma) \neq 0, l_S(\gamma) \geq Ci(\gamma,\alpha)|\log l_S(\alpha)|$. Then Lemma 4.2 gives:

Lemma 4.3 *Let L be a positive constant and let T be a positive constant $\leq |2\log L|$. If $l_S(\alpha) \leq L$, then there is a constant C depending on L such that*

$$d_{ls}(S, S_t) \leq \frac{|t|}{2C|\log l_S(\alpha)|}.$$

Combining Proposition 4.1 and Lemma 4.3, we have

Theorem 4.4 *Let L be a positive constant and let T be a positive constant $\leq |2\log L|$. If $l_S(\alpha) \leq L$ and $0 < |t| < T$, then there exists a constant C depending on L and T, such that*

$$\frac{d_{qc}(S, S_t)}{d_{ls}(S, S_t)} \geq C|\log l_S(\alpha)|.$$

As an application, we show

Theorem 4.5 *If S_0 is a conformal surface of infinite topological type with a pair of pants decomposition $\mathcal{P} = \{C_i\}$ such that there is a subsequence of $\{C_{i_k}\}$ contained in the interior of S_0 whose hyperbolic lengths tend to zero, then the identity map between the Teichmüller space $(\mathcal{T}_{qc}(S_0), d_{qc})$ and its image in $(\mathcal{T}_{ls}(S_0), d_{qc})$ is not locally bi-Lipschitz.*

Proof By assumption, there is a subsequence $\{C_{i_k}\}$ of elements of $\mathcal{P}$ with hyperbolic length $l_{S_0}(C_{i_k}) = \epsilon_k \to 0$. For any fixed t, let $S_{k,t}$ be the hyperbolic surface obtained by the time-t Fenchel-Nielsen twist deformation of S_0 along C_{i_k}. By Theorem 4.4, there is a constant C, depending on the maximum of ϵ_k and $|t|$, such that

$$\frac{d_{qc}(S_0, S_{k,t})}{d_{ls}(S_0, S_{k,t})} \geq C\,|\log \epsilon_k|\,.$$

Since $\log \epsilon_k \to \infty$ as $\epsilon_0 \to 0$, we have

$$\lim_{k\to\infty} \frac{d_{qc}(S_0, S_{k,t})}{d_{ls}(S_0, S_{k,t})} = \infty.$$

To see that the identity map between d_{qc} and d_{ls} is not locally bi-Lipschitz, we reason by contradiction. Assume there are constants C_1, C_2, such that for any $S \in \mathcal{T}_{qc}(S_0)$, if $d_{ls}(S_0, S) \leq C_1$, then $d_{qc}(S_0, S) \leq C_2 d_{ls}(S_0, S)$.

Consider $S_{k,t}$ as above, and note that the Teichmüller distance is controlled by t. In fact, if $|t| < T$ and $l_{S_0}(C_{i_k}) \leq L$, we have

$$d_{qc}(S, S_{k,t}) \leq C|t|,$$

where C is a constant depending on T and L. See [2, Lemma 8.3] for the proof. As a result, for any k, we can choose $|t|$ sufficiently small such that $d_{ls}(S_0, S_{k,t}) \leq C_1$. However, we have shown that as $k \to \infty$,

$$\frac{d_{qc}(S_0, S_{k,t})}{d_{ls}(S_0, S_{k,t})} \to \infty,$$

which contradicts the assumption that $d_{qc}(S_0, S) \leq C_2 d_{ls}(S_0, S)$. □

The following is an analogous result, with a sequence $\{S_k\}$ in $\mathcal{T}_{qc}(S_0)$, such that $d_{qc}(S_0, S_k) \to \infty$, while $d_{ls}(S_0, S_k) \to 0$.

Example 4.6 Let S_0 be a conformal structure of infinite type with pants-decomposition $\mathcal{P} = \{C_i\}$, such that there is a subsequence of $\{C_i\}$, contained in the interior of S_0, with hyperbolic length $l_{S_0}(C_{i_k}) = \epsilon_k$ tending to zero. Let S_k be the hyperbolic surface obtained by time-t_k Fenchel-Nielsen twist deformation of S_0 along C_{i_k}. Here $\{t_k\}$ is sequence of positive constants tending to infinity and satisfying $\frac{t_k}{|\log \epsilon_k|} \to 0$. Then it follows from the proof of Lemma 4.2 that $d_{ls}(S_0, S_k) \leq \frac{t_k}{2C|\log \epsilon_k|} \to 0$. On the other hand, the fact that $d_{qc}(S_0, S_k) \to \infty$ follows from Lemma 7.2 in [2].

5 The case of surfaces of finite type

In this section, we consider a hyperbolic surface $S = S_{g,m,n}$ of finite topological type, of genus g with m punctures and n boundary components and of negative Euler characteristic. It follows from our assumptions that when we equip such a surface with a conformal or a hyperbolic structure, then around each puncture, S has a neighborhood which is conformally equivalent to a punctured disk, and around each boundary component, S has a neighborhood which is conformally equivalent to an annulus. It is known that in this finite-type case we have the set-theoretic equalities $\mathcal{T}_{qc}(S) = \mathcal{T}_{ls}(S) = \mathcal{T}_{FN}(S)$, and we shall simply denote the Teichmüller space of S by $\mathcal{T}(S)$ unless a particular metric has to be specified.

The reader will notice that Proposition 4.1, Lemmas 4.2 and 4.3 and Theorem 4.4 are valid for any Riemann surface, whether it has finite or infinite topological type.

From Theorem 4.4, we deduce the following, by varying the length of the curve α.

Corollary 5.1 *For any Riemann surface of finite type of negative Euler characteristic which is not homeomorphic to a pair of pants, the identity map between the Teichmüller and the length spectrum metrics on $\mathcal{T}(S)$ is not a quasi-isometry.*

Remark 5.2 In the case where the surface is a pair of pants, although Corollary 5.1 cannot be deduced from Theorem 4.4, it is also true. Let us see this on the Teichmüller space of a surface of genus 0 with three boundary components; for the pairs of pants with one or two geodesic boundary components, we can do a similar reasoning. (For pair of pants with no geodesic boundary components the corollary is false since the Teichmüller space is reduced to a point.) Let us take an element S in the Teichmüller space represented by a hyperbolic pair of pants with three boundary components of length equal to 1, and for each $n \geq 2$, let us take an element represented by a surface $S_n \in \mathcal{T}(S)$ with three boundary components of length equal to n. Then we have

$$d_{ls}(S, S_n) = \frac{\log n}{2}.$$

Note that any geodesic arc connecting perpendicularly two boundary components of S_n has length approximately $2e^{-n}$. By reasoning on the doubles of S and S_n, we see that

$$d_{qc}(S, S_n) \geq \frac{n}{2}.$$

This shows that d_{qc} and d_{ls} are not quasi-isometric.

In the same case (of a surface a homeomorphic to a pair of pants), it follows for instance from Bishop's computations in [6] that the identity map between the Teichmüller metric and the length-spectrum metric is locally bi-Lipschitz. Note also that in this case the length-spectrum distance and the Fenchel-Nielsen distances are the same.

The result of Corollary 5.1, for surfaces of finite conformal type (that is, without boundary) was obtained independently and by other methods by Choi and Rafi in [7] and by Liu et al. in [15]. The result for surfaces of infinite topological type was obtained by Liu and Papadopoulos in [12]. The result for finite type surfaces with boundary is new.

To state other results for surfaces of finite type, we recall the definition of a metric that we introduced in [13] on the Teichmüller space of such a surface. The definition of this metric uses the set of homotopy classes of arcs on S. Let us give the precise definition.

An *arc* in S is the homeomorphic image of a closed interval whose interior is in the interior of S and whose endpoints are on the boundary of S. All homotopies of arcs that we consider are relative to ∂S, that is, they leave the endpoints of arcs on the set ∂S. An arc is said to be *essential* if it is not homotopic (relative to ∂S) to a map whose image is in ∂S.

We let $\mathcal{B} = \mathcal{B}(S)$ be the union of the set of homotopy classes of essential arcs on S with the set of homotopy classes of simple closed curves which are homotopic to boundary components.

Given an element γ of $\mathcal{B}$ and an element X of the Teichmüller space $\mathcal{T}(S)$, the *length* of γ with respect to X, denoted by $l_X(\gamma)$ is defined, in analogy with the length of an element of $\mathcal{S}$, as the length of the unique geodesic arc homotopic to γ in a hyperbolic metric representing X.

In [13] and [14] we studied the following metric on $\mathcal{T}(S_{g,m,n})$. For X and Y in this space, we set

$$\delta_L(X, Y) = \log \max\left(\sup_{\gamma \in \mathcal{S}\cup\mathcal{B}} \frac{l_Y(\gamma)}{l_X(\gamma)}, \sup_{\gamma \in \mathcal{S}\cup\mathcal{B}} \frac{l_X(\gamma)}{l_Y(\gamma)} \right). \tag{13}$$

We showed that this function δ_L defines a metric, and that this metric is also given by

$$\delta_L(X, Y) = \log \max\left(\sup_{\gamma \in \mathcal{B}} \frac{l_Y(\gamma)}{l_X(\gamma)}, \sup_{\gamma \in \mathcal{B}} \frac{l_X(\gamma)}{l_Y(\gamma)} \right). \tag{14}$$

We call δ_L the *arc metric* on the Teichmüller space of the surface with boundary.

Any hyperbolic surface of finite type $S_{g,m,n}$ obviously satisfies Shiga's Condition (2), and Theorem 3.5 applies to such a surface. Let L be an upper bound for the hyperbolic length of the boundary geodesics of $S_{g,m,n}$. A result by Bers [5] says that there exists a pants decomposition of S with an upper bound L_0 for the lengths of the decomposition curves, with L_0 depending only on g, m, n and L.

We shall use the following classical terminology.

Given a positive real number ϵ, the ϵ-thick part of the Teichmüller space of S, denoted by $\mathcal{T}_\epsilon(S)$, is defined as

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \forall \gamma \in \mathcal{S}, l_X(\gamma) \geq \epsilon\}.$$

We let $\mathcal{D}$ be the set of boundary components of S. We shall use the following terminology that was introduced in [14].

For $\epsilon > 0$ and $\epsilon_0 \geq \epsilon$, the ϵ_0-*relative* ϵ-thick part of Teichmüller space, denoted by $\mathcal{T}_{\epsilon,\epsilon_0}$, is the subset of the ϵ-thick part of Teichmüller space defined as

$$\mathcal{T}_{\epsilon,\epsilon_0}(S) = \{X \in \mathcal{T}(S) \mid \forall\gamma \in \mathcal{S}, l_X(\gamma) \geq \epsilon \text{ and } \forall\delta \in \mathcal{D}, l_X(\delta) \leq \epsilon_0\}.$$

We prove the following:

Theorem 5.3 *Let S be a topologically finite type surface. For any $\epsilon > 0$ and any $\epsilon_0 \geq \epsilon$, the identity map between any two of the three metrics d_{ls}, d_{qc} and δ_L on $\mathcal{T}_{\epsilon,\epsilon_0}(S)$ is globally bi-Lipschitz, with a bi-Lipschitz constant depending on the topological type of S, on ϵ and on ϵ_0.*

Proof We first prove that the identity map

$$\mathrm{Id} : (\mathcal{T}_{\epsilon,\epsilon_0}, d_{ls}) \to (\mathcal{T}_{\epsilon,\epsilon_0}, d_{qc})$$

is globally bi-Lipschitz. More precisely, we prove that for any $X, Y \in \mathcal{T}_{\epsilon,L}$, we have

$$d_{ls}(X, Y) \leq d_{qc}(X, Y) \leq K d_{ls}(X, Y). \tag{15}$$

where K depends on the topological type of S, ϵ and ϵ_0.

The left hand side inequality in (15) follows from Wolpert's lemma.

From Theorem 1.4 applied to surfaces of topological finite type, for any $D > 0$, if $X, Y \in \mathcal{T}_{\epsilon,L}$ with $d_{ls}(X, Y) \leq D$, we have $d_{qc}(X, Y) \leq C d_{ls}(X, Y)$, where C depends on D, on the topological type of S, on ϵ and on ϵ_0. Therefore, if $d_{ls}(X, Y) \leq D$, the right hand side inequality of (15) is satisfied. (We could take, for example, $D = 1$.)

Now assume that $d_{ls}(X, Y) \geq D$. From (12) of Theorem 6.3 in [13], $d_{qc}(X, Y) \leq \delta_L(X, Y) + D$. From Theorem 3.6 in [14], $\delta_L(X, Y) \leq d_L(X, Y) + K$. Thus, we have $d_{qc}(X, Y) \leq d_{ls}(X, Y) + D + K$. This gives

$$d_{qc}(X, Y) \leq d_{ls}(X, Y) + K_1,$$

where K_1 is a constant depending only on the topological type of S, on ϵ and on ϵ_0.

As $d_{ls}(X, Y) \geq D$, we have

$$K_1 = \frac{K_1}{D} D \leq \frac{K_1}{D} d_{ls}(X, Y),$$

and

$$d_{qc}(X, Y) \leq (1 + \frac{K_1}{D}) d_{ls}(X, Y).$$

This proves the right hand side inequality of (15) in all cases.

It remains to show that the identity map

$$\mathrm{Id} : (\mathcal{T}_{\epsilon,\epsilon_0}, d_{ls}) \to (\mathcal{T}_{\epsilon,\epsilon_0}, \delta_L)$$

is globally bi-Lipschitz. We use results proved in [13] and [14] on the natural embeddings between the Teichmüller space $\mathcal{T}(S)$ and the Teichmüller space $\mathcal{T}(S^d)$ of the double S^d of S. From the proof of Theorem 3.3 of [13], this embedding is distance-preserving for the quasiconformal metrics on the two spaces. From Corollary 2.8 of [14], this embedding is distance-preserving with respect to the metric δ_L on $\mathcal{T}(S)$ and d_{ls} on $\mathcal{T}(S^d)$. Furthermore, Proposition 4.2 of [13] shows that the natural embedding $\mathcal{T}(S) \to \mathcal{T}(S^d)$ sends an ϵ_0-relative

ϵ-thick part of $\mathcal{T}(S)$ to an ϵ'-thick part of $\mathcal{T}(S^d)$. We already showed that on such an ϵ'-thick part of $\mathcal{T}(S^d)$, the identity map between the quasiconformal and the length spectrum metrics is globally bi-Lipschitz. Therefore, the identity map between the quasiconformal metric and the arc metric δ_L on the ϵ_0-relative ϵ-thick part of $\mathcal{T}(S)$ is globally bi-Lipschitz. □

In the case of a surface S of finite type, we define more simply, for $\epsilon > 0$, the ϵ-thick part of $\mathcal{T}(S)$, denoted by $\mathcal{T}_\epsilon(S)$, as

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \forall \gamma \in \mathcal{S}, l_X(\gamma) \geq \epsilon\}.$$

We have the following theorem, analogous to Theorem 5.3. The proof is similar to the proof of the first part of Theorem 5.3.

Theorem 5.4 *Let S be a topologically finite type surface without boundary. For any $\epsilon > 0$, the identity map between the two metrics d_{ls}, d_{qc} on $\mathcal{T}_\epsilon(S)$ is globally bi-Lipschitz.*

Note that the Fenchel-Nielsen distance is not included in Theorems 5.3 and 5.4.

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